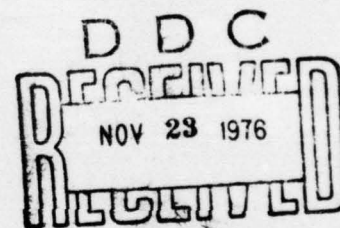# NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

A PRACTICAL COMPUTATION METHOD FOR

STEADY STATE SOLUTION OF $M/E_k/c$ QUEUE

by

Mehmet Celayir

September 1976

Thesis Advisor: R. R. Read

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REPORT DOCUMENTATION PAGE

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BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)	5. TYPE OF REPORT & PERIOD COVERED
A Practical Computation Method for Steady State Solution of M/E$_k$/c Queue.	Master's Thesis, September 1976
	6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)	8. CONTRACT OR GRANT NUMBER(s)
Mehmet Celayir	

9. PERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Naval Postgraduate School Monterey, California 93940	

11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE
Naval Postgraduate School Monterey, California 93940	September 1976
	13. NUMBER OF PAGES 56

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report)
Naval Postgraduate School Monterey, California 93940	Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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S/N 0102-014-6601

A Practical Computation Method for

Steady State Solution of $M/E_k/c$ Queue

by

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Submitted in partial fulfillment of the
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MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
September 1976

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3

ABSTRACT

Poisson arrivals and Erlang service time type

Numerical tables available for $M/E_k/c$ queueing systems are discussed. A new approximation method for steady–state information and waiting time distribution of this queueing system was developed. Validity of approximation was established directly for the large waiting times and by simulation for the smaller values. The developed method enables one to find delay probability, expected number in the queue and in the system, expected time to be spent in the queue and in the system, and probability of waiting for more than a specified time t.

TABLE OF CONTENTS

ACKNOWLEDGEMENT

8

I. INTRODUCTION AND SUMMARY

Multi—server queues constitute a large proportion of queueing systems which arise in practice. Among those, queues with Poisson arrivals and Erlang service times (which will be denoted as $M/E_k/c$) occupy an important position, as seen in the banks, airport check—in counters, hotels, super-markets etc. However, no significant theoretical work was done until the late 60's. In recent years, methods for the steady state solution of $M/E_k/c$ queues became available, fol-lowed by the numerical tables which were obtained by imple-mentation of these methods. Even so, there is still some computational difficulty involved obtaining some particular information about above—mentioned queues, such as probabili-ty of delay exceeding a specified time length.

The importance of the $M/E_k/c$ queue is due to the fact that it is used to model any queueing system whose service time distribution is believed to be unimodal. The solutions are known for extreme values of k: exponential service time distribution for k=1 and constant service times as $k \to \infty$. Usable solutions for multi-server queues having either expo-nential or constant service times are readily available. The $M/E_k/c$ queue is also important by providing a large variety of service time distributions in between these two extremes.

The purpose of this study is, by means of computer simulation and examination of numerical tables published earlier, to present a simple and accurate computation method for obtaining information about steady-state solution of the $M/E_k/c$ queue.

For this purpose, a general description of $M/E_k/c$ queueing model with definitions and assumptions for the analysis of it are given in Chapter II, followed by a summary of former studies.

In Chapter III, the numerical tables are analyzed and some conclusions are drawn in terms of a simple form approximation for steady-state distribution of waiting times.

This approximation for the distribution of waiting times needed verification for small values of them. These cases were studied by computer simulation. The procedure and the results of simulation are given in Chapter IV.

The results of the analysis of numerical tables and simulation are combined and generalized in Chapter V. Then the computational method developed by this generalization is described on a step-by-step basis. With this method, there is no need for numerical tables, computations are very simple, and the results are accurate for most practical purposes (the error being about 2% in general). To demonstrate the advantages of the proposed method, a comparison of it with the other method which uses numerical tables is given at the end of that chapter.

II. The $M/E_k/c$ Queue

The description of the $M/E_k/c$ queueing model with assumptions, definition of parameters and system variables along with some basic definitions in the queueing theory forms the first half of this chapter. A brief discussion of the previous studies on this model then follows.

A. DESCRIPTION OF THE SYSTEM

The $M/E_k/c$ queue is a multi-server queueing system with c servers, where arrivals occur according to a Poisson process with rate λ, and service times have Erlang distribution with shape parameter k. It is also an infinite queue, i.e. there is no limitation for the number of customers in the system.

The distribution of interarrival times X is given by

$$f_X(x) = \lambda e^{-\lambda x} , \quad x \geq 0$$

and the service times have the following probability density function:

$$f_Y(y) = \frac{1}{(k-1)!\beta^k} \, y^{k-1} e^{-y/\beta} , \quad y \geq 0$$

with mean $k\beta$ and variance $k\beta^2$.

1. Definitions of Parameters

a. Arrival Rate and Service Rate

Arrival rate to the system is the reciprocal of mean interarrival time and denoted by λ. Similarly, service rate of a server is defined as reciprocal of mean service time and represented by μ. Then, $\mu = \frac{1}{k\beta}$.

b. Traffic Intensity

Traffic intensity ρ is defined by the following expression:

$$\rho = \lambda/c\mu .$$

c. Offered Load

Offered load is the ratio of arrival rate λ to service rate μ and denoted by α. It can also be expressed by product of number of servers c and traffic intensity ρ as follows

$$\alpha = c\rho .$$

d. Coefficient of Variation

The ratio of standard deviation to mean is defined as coefficient of variation of a probability distribution. It is denoted by V. For Erlang distribution coefficient of variation is expressed as

$$V = \frac{\sqrt{k\beta^2}}{k\beta} = \frac{1}{\sqrt{k}} .$$

2. System Variables

a. Number in the System

The total of the number of customers waiting in the queue and those in the service is defined as number in the system and denoted by N. The number of customers only in the queue is represented by Nq. Expected values of N and Nq are denoted by L (average number in the system) and Lq (average number in the queue), respectively.

b. Waiting Time

The time spent in the queue by a customer is defined to be the waiting time of a customer. The time spent in the system is then the sum of waiting time and service time of a customer. Tq denotes waiting time and T denotes the time spent in the system. Expected values of T and Tq are denoted by W (average time in the system) and Wq (average waiting time) respectively.

c. State Probabilities

The number in the system at a particular time is defined as the state of the system. Then state probability $P(N = n|t)$ is the probability of the system being in state n at time t.

d. Delay Probability

Delay probability is the probability that the arriving customer finds all the servers busy and enters the waiting line. It is denoted by $P(Tq > 0)$. Also, by definition it follows that

$$P(Tq > 0) = P(N \geq c).$$

3. The Steady–State Solution

The steady–state solution is defined to be the probability distribution of the number in the system when the system achieves statistical equilibrium. In the steady–state, the state probabilities do not depend on t, i.e. $P(N = n|t) = P(N = n)$. Also it follows that

$$\sum_{n=0}^{\infty} P(N = n) = 1 .$$

In queueing theory, it is a well–known fact that [1] the steady–state is achieved if and only if the traffic intensity ρ is less than 1.

4. Assumptions

The $M/E_k/c$ queueing system under study is assumed to have the following properties:

(i) There is only one waiting line no matter how many servers there are.

(ii) The customer at the head of the waiting line starts getting service immediately whenever a server becomes available.

(iii) Order of service is first–come first–served.

(iv) If an arriving customer finds all servers busy, he joins the waiting line. The waiting line has unlimited capacity.

(v) The service channels (servers) are indexed consecutively. If an arriving customer finds more than one

server vacant, he goes to the server with smallest index
(this assumption doesn't cause any loss of generality, but
is convenient for the computer simulation model).

(vi) The servers are homogeneous, i.e. the service
distribution is the same for all servers.

(vii) Only one customer at a time can be served by
a server.

B. FORMER STUDIES

There are numerous studies in the literature of queueing
theory, done for the steady-state solution of $M/E_k/c$ queue.
But most of them cover only some particular values of k and
c, and their results cannot be generalized. Therefore, such
studies will not be mentioned here individually.

The earliest suggestion known by the author about the
solution of general $M/E_k/c$ system was mentioned by Lee [2].
He stated referring to a case study done in 1956 that mean
waiting time can be approximated by use of David Owen's sug-
gestion. The formula given in [2] was said to be usable for
$0.7 \leq V \leq 1.0$ and as follows,

$$W_q = \frac{1}{2} W_{q1} (1+V^2)$$

where
W_q : mean waiting time for $M/E_k/c$ queue

W_{q1}: mean waiting time for M/M/c queue

V : coefficient of variation of Erlang (k)
distribution, $V^2 = 1/k$.

The validity of the approximation was demonstrated by
comparing its results with simulation results for different

cases. However, the approximation is just for mean waiting time or queue length, hence it isn't possible to approximate state probabilities. Also, no reference was given for details of David Owen's suggestion. This approximation will be mentioned again later in this study.

The steady-state solution of the general $M/E_k/c$ queue was first studied by Mayhugh and McCormick [3]. The results can be applied to the cases with any value of k and c. However, the computation procedure is so complex that it requires a very considerable amount of work.

A parallel study was also done by Heffer [4]. The solution method proposed by Heffer differs from Mayhugh and McCormick's, but it also is very complex.

The reader is referred to the references [5] and [6] for more detailed discussion of these two studies.

The most recent study was done by Yu [5]. It concerned the $E_m/E_k/c$ queue with heterogeneous servers, but results were also stated for homogeneous server case. Then letting m = 1 gives the solution procedure for $M/E_k/c$ queue. However, like the other two studies, the computations required still demand an enormous amount of work.

The theoretical results obtained by Heffer [4] and Yu [5] formed the basis of the only numerical tables available for $M/E_k/c$ queue, prepared by Hillier and Lo [6]. The tables have cases of $M/E_k/c$ queue with limited values of k and c ($k \leq 8$, $c \leq 10$), and a few cases for $E_m/E_k/c$ queue. These tables will be discussed in detail in the following chapter.

III.   DISCUSSION OF NUMERICAL TABLES AND
       DEVELOPING A NEW COMPUTATION METHOD

In this chapter, a detailed review and discussion of the numerical tables given in [6] will be made. Then, using the results of these discussions, a new computation method will be developed.

Sample pages from the numerical tables are given in Appendix B.

A.   DISCUSSION OF NUMERICAL TABLES

1.   General Description

The tables under consideration were designed for general $E_m/E_k/c$ systems. The analysis however will be focused on the cases in which m = 1 (i.e. $M/E_k/c$ system), which forms a large porportion of the tables. As mentioned earlier, the values of k and c for various tables are as follows:

$$k=2; \quad c=1,2,\ldots, 10$$
$$k=3; \quad c=1,2\ldots., \quad 5$$
$$k=4; \quad c=2,3$$
$$k=5,6,\ldots,8; \quad c=2$$

The information given in each table is:

(i)   State probabilities, $P(N=n)$, and

(ii)   Cumulative state probabilities, $P(N \leq n)$, for
n=1,2,...;

(iii) Delay probability, $P(T_q>0)$;

(iv)   Expected number in the system, L;

17

(v)   Average queue length, Lq;

(vi)   Average queue length for the equivalent M/M/c system with the same arrival and service rates, L_{q1}.

(vii) The ratio L_q/L_{q1}.

All this information is given in each case and for the following values of traffic intensity ρ :

ρ = 0.10, 0.20,..., 0.50, 0.55,..., 0.95, 0.98, 0.99.

So, all the information is given for steady-state conditions.

Average queue length L_{q1} for the equivalent M/M/c system is computed by using the following formula [7],

$$L_{q1} = \frac{(c\rho)^c}{c!\,(1-\rho)^2}\, P_o \qquad\qquad (3.1)$$

where

$$\frac{1}{P_o} = \sum_{j=0}^{c-1} \frac{(c\rho)^j}{j!} + \frac{(c\rho)^c}{c!\,(1-\rho)}$$

The rest of the information given in the tables was computed by the results of theoretical work mentioned in the last chapter, done by Heffer and Yu.

2. Use of Tables

The kind of information listed above can be obtained directly from the tables. However, by making use of some general relationships in the queueing theory, some other information can be obtained besides those mentioned above.

a. Interpolation

No interpolation method is specified in the explanation sections for the tables in [6], for the values of

18

ρ different from those given above. The default position taken is linear interpolation.

    b. Waiting Times

    The average waiting time W_q and average time in the system W are frequently of interest in the analysis of queueing systems. These values can be obtained from the tabulated values of L_q and L respectively, by using the following relationships [8],

$$W_q = \frac{L_q}{\lambda} , \qquad\qquad W = \frac{L}{\lambda} .$$

    The probability $P(T_q > t)$ that waiting time exceeding t is also important in the analysis of queueing systems. Although this kind of information is not given in the tables, $P(T_q > t)$ can be approximated from the state probabilities, $P(N=n)$, as follows:

$$P(T_q > t) = \sum_{n=c}^{\infty} P(N=n) \; P\{D \le [k(n-c+1)-1]\}, \; t \ge 0,$$

$$(3.2)$$

where D is a Poisson random variable with mean $ck\mu t$. The reader is referred to the discussion in [6] for the stochastic reasoning of this relationship. The quantity $P\{D \le [k(n-c+1)-1]\}$ can be obtained from Poisson distribution tables with mean $ck\mu t$. If this mean exceeds 25, then the

normal distribution with the same mean and variance can be
used as an approximation to Poisson distribution.

3. Discussion

    a. Delay Probabilities

        Similar to $M/E_k/c$, $M/M/c$ denotes a multi-server
queueing system with Poisson arrivals of rate λ and exponen-
tial service times of service rate μ. The delay probabili-
ties for $M/M/c$ queue can be computed by using the formula
[7],

$$P(T_q>0) = \frac{(c\rho)^c}{c!\,(1-\rho)} \frac{1}{\left(\sum_{j=0}^{c=1} \frac{(c\rho)^j}{j!} + \frac{(c\rho)^c}{c!\,(1-\rho)}\right)} \qquad (3.3)$$

        A chart is also available in Appendix A for $c=2$,
$3,\ldots,15$ and $0<\rho<1$, for $P(T_q>0)$.

        A comparison of delay probabilities of $M/E_k/c$
and $M/M/c$ systems for selected values of ρ, c and k is given
in Table I of this study. A careful examination of this
table shows that the corresponding delay probabilities of
the two systems for the same value of ρ are very close.
Since the formula or charts are available for $M/M/c$ system,
this comparison shows that they can also be used to estimate
delay probabilities for $M/E_k/c$ system with a very small
error, usually less than 0.01.

20

TABLE I

Comparison of Delay Probabilities in Different Systems

ρ	c=2				c=10 →			c=20	
	k=1	k=2	k=8	k=∞	k=1	k=2	k=∞	k=1	k=∞
0.10	0.0182	0.0181	0.0179	0.0178	$\sim 10^{-6}$	$\sim 10^{-6}$			
0.30	0.1385	0.1373	0.1352	0.1336	0.0012	0.0011	0.0011	$\sim 10^{-7}$	$\sim 10^{-7}$
0.50	0.3333	0.3308	0.3265	0.3232	0.0361	0.0352	0.0331	0.0037	0.0035
0.60	0.4499	0.4471	0.4423	0.4387	0.1013	0.0987	0.0925	0.0241	0.0220
0.70	0.5765	0.5736	0.5688	0.5653	0.2217	0.2167	0.2043	0.0936	0.0844
0.80	0.7111	0.7087	0.7047	0.7018	0.4092	0.4021	0.3847	0.2561	0.2345
0.90	0.8526	0.8512	0.8488		0.6687	0.6625			
0.95	0.9256	0.9248	0.9235		0.8256	0.8216			

k=1: M/M/c system

k=∞: M/D/c system

k>1: M/E$_k$/c system

21

b. Average Queue Length and Number in the System

A careful examination of the "Ratio" column of Table I in [6] indicates that

$$\lim_{\rho \to 1} \frac{L_q}{L_{q1}} = \frac{k+1}{2k} = \frac{1}{2}(1 + \frac{1}{k}) = \frac{1}{2}(1 + V^2) \qquad (3.4)$$

which is essentially the same as Owen's suggestion [2]. However, it works beyond the limits of coefficient of variation V that were stated by Owen. Also (3.4) is true even when ρ is markedly less than one. The theoretical verification for (3.4) was developed by Yu in [9].

A more precise approximation formula was also developed by Hillier and Lo in [6] which provides more accuracy for small values of ρ. In this case, the approximation formula is given by

$$L_q \approx \frac{1}{2}(1+g) \; (1 + \frac{1}{k}) \; L_{q1} \qquad (3.5)$$

where $g = f(\rho,k,c)$. Then the expression for g has been obtained by linear regression. The exact coefficients for $g(\rho,k,c)$ are given in [6]. However, a rougher expression which is more convenient for computations will be used here, given as

$$g = \frac{1}{12}(\frac{k-1}{k+1}) \; (c-1)^{2/3} \; \{(1-\rho) + (1-\rho)^2\} \qquad (3.6)$$

22

The main purpose of Hillier and Lo for introducing g into approximation formula (3.5) was to provide means of approximation for the cases with larger k and c values which were not covered in [6], namely, extrapolation for larger values of c and k. One way to check the validity of the computational formula for g is to first let k go to infinity, then compare the values of average waiting time W_q computed by the formula given below with average waiting time W_{qD} of M/D/c queue. This latter queue has Poisson arrivals with rate λ and constant service times of length $1/\mu$ (or $k\beta$), and one can use the fact that the distribution of service times can be approximated by constant service time distribution for very large k as mentioned in Chapter I.

The computation formula for W_{qD} of M/D/c queue is given in [2] as follows,

$$W_{qD} = \sum_{i=1}^{\infty} e^{-i\alpha} \left[\sum_{j=ic}^{\infty} \frac{(i\alpha)^j}{j!} - \frac{1}{\rho} \sum_{j=ic+1}^{\infty} \frac{(i\alpha)^j}{j!} \right]$$

However, a more convenient form for computations can be obtained as

$$W_{qD} = \sum_{i=1}^{\infty} \left[\frac{e^{-i\alpha}(i\alpha)^{ic}}{(ic)!} + (1 - \frac{1}{\rho}) \sum_{j=ic+1}^{\infty} \frac{e^{-i\alpha}(i\alpha)^j}{j!} \right]$$

$$(3.7)$$

23

which permits the use of Poisson distribution tables. But it still requies a great deal of computational work. Fortunately, a very convenient chart was developed by Shelton [10] for W_{qD} for $1 \leq c \leq 100$ and $0.10 \leq \rho \leq 0.96$.

The comparison mentioned above can be made having the necessary tools available and Table II can be used for this purpose. The values of W_{qD} for M/D/c system were obtained from Shelton's charts. Average waiting time W_q for M/E_kc system is computed by using the following formula,

$$W_q = \frac{1}{2\lambda}\left[1 + \frac{1}{12}(\frac{k-1}{k+1})(c-1)^{2/3}\{(1-\rho) + (1-\rho)^2\}\right](1+ \frac{1}{k})L_{q1}$$

where L_{q1} is computed from (3.1). If the expression given by (3.6) is true for very large values of k, then it should also be true that

$$\lim_{k\to\infty} W_q = W_{qD}$$

Taking the limit of W_q gives

$$\lim_{k\to\infty} W_q = \frac{1}{2\lambda}\left[1 + \frac{1}{12}(c-1)^{2/3}\{(1-\rho) + (1-\rho)^2\}\right] L_{q1}$$

On the other hand, if g is omitted, the limit will then be

$$\lim_{k\to\infty} W_q = \frac{1}{2\lambda} L_{q1} = \frac{1}{2} W_{q1}$$

24

A comparison of W_{qD} and these two limits for eight different cases is given in Table II. Investigation

TABLE II.

Comparison of W_{qD} and Limits of W_q With or Without g

Case No	c	ρ	$\frac{1}{2} W_{ql}$	$W_q (k=\infty)$	W_{qD}
1	2	0.80	1.185	1.209	1.08
2	2	0.80	1.422	1.451	1.296
3	5	0.90	2.287	2.34	2.16
4	5	0.90	1.144	1.17	1.08
5	10	0.90	1.337	1.391	1.36
6	10	0.90	1.505	1.563	1.53
7	20	0.90	3.305	3.766	3.36
8	20	0.90	5.508	5.88	5.6

of these results shows that, for almost all the cases the limit with g omitted is closer to W_{qD} than the other limit. This indicates that the formula (3.6) should be reviewed for large k values. However, this revision can be done only after some additional tables become available for the cases not covered in [6].

c. Waiting Times

The method of approximating the probability $P(T_q > t)$ that the waiting time will exceed t using the state probabilities $P(N=n)$ was described earlier in the chapter. However, this method assumes that an arriving customer will

25

have to wait for approximately $k(n-c+1)$ service phase (exponentially distributed with mean β) completions before a server becomes available to start his service. It is stated in [6] that this approximation for $P(T_q>t)$ should be better for larger values of ρ and t due to this assumption. It should be added that, for large k and small t, the approximation fails greatly in representing the actual distribution of waiting times for the same reason.

B. NEW COMPUTATION METHOD

The construction of a practical computation method using the facts obtained from the preceding discussions will form the rest of the chapter. This new method is desired to be applicable to the queueing problems involving $M/E_k/c$ systems met in practice, with simple and quick computational work without any need for tables, interpolations etc.

The kinds of information which are desired to be able to compute by the new method for $M/E_k/c$ queueing system are mainly

(i)   Delay probability,

(ii) Average waiting time, average time in the system, average queue length and average number in the system,

(iii) Probability distribution of delay times and probability that waiting time will exceed a specified time length.

1. Delay Probability

In the discussion earlier it was pointed out that the delay probabilities $P(T_q>0)$ for the systems $M/E_k/c$ and

M/M/c are very close for the same value of ρ. An examination of the comparisons given in Table I shows that the delay probabilities are very close for M/M/c (k=1), M/E$_k$/c (k>1) and M/D/c (k=∞) systems with the same value of ρ. The differences between the delay probabilities for the same ρ are less than 0.01 for almost all the cases and gets even smaller for ρ close to one. This comparison suggests that delay probabilities of the M/M/c system can be used for corresponding M/E$_k$/c system also. These delay probabilities are computed by using formula (3.3). The formula is not suitable for large values of c, however, and a chart is given in Appendix A for up to about 16. For greater values of c, the charts are also available in [10] and [11].

2. Average Queue Length and Average Waiting Time

In the previous discussion, after examination of the tables in [6] it was concluded that the average queue length for M/E$_k$/c system can be approximated by

$$L_q = \frac{1}{2}(1+g) \; (1 + \frac{1}{k})L_{q1} \qquad (3.8)$$

where L_{q1} is computed according to (3.1), and g is computed by

$$g = \frac{2}{12} \; \frac{k-1}{k+1}(c-1)^{2/3} \{(1-\rho) + (1-\rho)^2\} \; .$$

Contribution of g in equation (3.8) is negligible for practical purposes in most cases. Also it was shown earlier in this chapter that the formula g requires some

modifications. Therefore it will be omitted in further discussions.

Now, attention will be focused on equations (3.1) and (3.3). Rewriting equation (3.1) in a different form, then substituting (3.3) gives

$$L_{q1} = \frac{\rho}{(1-\rho)} \cdot \frac{(c\rho)^c}{c!(1-\rho)\left[\sum_{j=0}^{c-1}\frac{(c\rho)^j}{j!} + \frac{(c\rho)^c}{c!(1-\rho)}\right]} \quad \text{and}$$

$$L_{q1} = \frac{\rho \cdot P(T_q>0)}{(1-\rho)}$$

Then it follows that

$$W_{q1} = \frac{L_{q1}}{\lambda} = \frac{P(T_q>0)}{c\mu(1-\rho)}.$$

Using (3.8) with g=0, some approximation formulas for average queue length and average waiting time are obtained respectively as follows

$$L_q = \frac{1}{2}\rho\frac{P(T_q>0)}{(1-\rho)}\left(1+\frac{1}{k}\right) \qquad (3.9)$$

$$W_q = \frac{1}{2}\frac{P(T_q>0)}{c\mu(1-\rho)}\left(1+\frac{1}{k}\right) \qquad (3.10)$$

where $P(T_q>0)$ is delay probability for M/M/c system which can be obtained from the charts very easily. Then it is very simple to compute L_q and W_q using equations (3.9) and (3.10).

The average number in the system L and average time in the system W can be computed by following well-known relationships of queueing theory,

$$L = L_q + \frac{\lambda}{\mu} , \qquad W = \frac{L}{\lambda} .$$

3. Waiting Time Distribution

Suppose that the conditional waiting time distribution given $T_q > 0$ for $M/E_k/c$ queueing system can be approximated by

$$P(T_q > t \mid T_q > 0) = e^{-bt} .$$

Then using Bayes' theorem it follows that

$$P(T_q > t) = P(T_q > t \mid T_q > 0) P(T_q > 0) = e^{-bt} P(T_q > 0) . \quad (3.11)$$

Let F_{Tq} be the CDF of waiting times. Then

$$F_{Tq}(t) = 1 - P(T_q > 0) e^{-bt} , \qquad t \geq 0.$$

Now average waiting time can be computed as follows

$$W_q = E(T_q) = {}_0\!\int^\infty (1 - F_{Tq}(t)) dt = {}_0\!\int^\infty P(T_q > 0) e^{-bt} dt$$

$$W_q = \frac{P(T_q > 0)}{b} . \qquad (3.12)$$

Suppose the parameter b is modeled as

$$b = \frac{2c\mu(1-\rho)}{(1+V^2)}$$

29

Then,

$$W_q = \frac{P(T_q > 0)}{2c\mu(1-\rho)} (1+V^2) \qquad (3.13)$$

where V is coefficient of variation of service time distribution. Notice that equation (3.13) is exactly the same as equation (3.10). However, there can exist some other waiting time distribution to give the same average waiting time as given by (3.13), so it must be shown that for $M/E_k/c$ system, the distribution of waiting times can be approximated by

$$F_{Tq}(t) = 1 - P(T_q > 0) e^{-\frac{2c\mu(1-\rho)t}{(1+V^2)}} \quad , \quad t \geq 0 \qquad (3.14)$$

If (3.14) is true, then $P(T_q > t)$ computed by (3.11) should be approximately equal to the one obtained by the procedure suggested by Hillier and Lo [6] which uses state probabilities as described earlier by equation (3.2). Table III gives the comparison of $P(T_q > t)$ values obtained by these two different formulas for various values of t. It should be recalled that approximation by state probabilities is poor for small values of t. However, for $t \geq W_q$ the two results agree very well. The difference is ease of computations. Equation (3.11) is much easier than equation (3.2) to compute the same value.

The validity of approximation (3.14) for small values of t will be shown by simulation, which forms the next chapter. Then the suggested computation method will formally be described on a step-by-step basis in last chapter.

TABLE III.

Comparison of the Two Approximation Methods

| System Parameters | | | | | $P(T_q > t)$ Probability that waiting time exceeds t | | | | | | |
c	k	ρ	W_q	Case	t=0.1	t=0.5	t=1.0	t=2.0	t=3.0	t=4.0	t=5.0
2	8	0.8	1.60	I	0.510	0.523	0.433	0.303	0.202	0.129	0.080
				II	0.680	0.569	0.456	0.290	0.187	0.120	0.077
2	8	0.9	3.84	I	0.726	0.716	0.646	0.544	0.437	0.352	0.279
				II	0.834	0.763	0.683	0.546	0.437	0.350	0.280
3	3	0.8	0.85	I	0.506	0.447	0.321	0.152	0.068	0.030	0.013
				II	0.600	0.445	0.307	0.145	0.061	0.032	0.015
4	2	0.9	1.47	I	0.688	0.585	0.454	0.299	0.159	0.087	0.052
				II	0.746	0.605	0.460	0.271	0.159	0.093	0.054
10	2	0.9	1.00	I	0.583	0.468	0.342	0.178	0.088	0.036	0.013
				II	0.625	0.479	0.343	0.176	0.090	0.046	0.023

I. Computed by the procedure with state probabilities, eq. (3.2)

II. Computed by using exponential approximation, eq. (3.11)

IV. COMPUTER SIMULATION

A computer simulation model for $M/E_k/c$ system is used to investigate the validity of approximation for waiting time distribution introduced in the preceding chapter. Validity of this approximation for moderate and large values of t was demonstrated in the same chapter. Therefore, analysis will now be focused on the approximation for small values of t.

A. COMPUTER PROGRAM

1. Model

Since the model used to construct the simulation program is based on the same assumptions as stated in Chapter II, it won't be described again in this chapter. The computer language selected was SIMSCRIPT which is especially convenient for simulation of multi-server queueing systems. The reader is referred to [12] and [13] for a brief idea about SIMSCRIPT if he is not familiar with this language. Main reference for SIMSCRIPT however is [14].

One way to check for small t the validity of approximation distribution developed earlier is to keep the frequency table of waiting times of the customers during the simulation. Let $M(t)$ be the number of customers with waiting time greater than t and M be total number of customers

served during simulation period. Then the estimator for probability of waiting time greater than t is

$$\widehat{P(T_q > t)} = \frac{M(t)}{M} \tag{4.1}$$

This value can be compared with the value given by the approximation formula to check its validity.

The computer program is given in Appendix B. It is written in the way that it would be possible to obtain the estimates of all kinds of information about the queueing system being simulated; average waiting time and average queue length, state probabilities, delay probability etc. However, only the estimates given by (4.1) will be discussed here.

2. Variance Reduction

One of the problems encountered in the simulation of queueing systems is the high variability of waiting times, especially when traffic intensity ρ is high. Also strong positive correlation between the waiting times of consecutive customers causes serious errors if the standard formula is used to estimate the sample variance of waiting times since this formula will underestimate the true variance [1].

There are several methods developed to overcome this difficulty with high variability. Interested reader can find comprehensive information in references [12], [15] and [16]. The variance reduction method which will be used here is antithetic variates.

33

In the simulation program, two different random number streams are used to generate interarrival and service times. Once a uniform variate between 0 and 1 is generated, then the random variate from a desired distribution can be obtained by

$$X = F_x^{-1}(u) \qquad\qquad (4.2)$$

where u is a uniform variate between 0 and 1 and F_x^{-1} denotes the inverse of CDF of X [12]. Then, for example, a sequence of exponentially distributed random variates with mean $1/\lambda$ can be obtained from a stream of uniform variates by using

$$X = \frac{-1}{\lambda} \ln(u) \qquad\qquad (4.3)$$

Let Z_1 and Z_2 be estimators of a parameter, obtained from two different simulation samples and $Z_3 = \frac{1}{2}(Z_1 + Z_2)$. Then

$$Var(Z_3) = \frac{1}{4} Var(Z_1 + Z_2) = \frac{1}{4} [Var(Z_1) + Var(Z_2) + Cov(Z_1, Z_2)]$$

$$(4.4)$$

which implies that maximum negative correlation between Z_1 and Z_2 would minimize $Var(Z_3)$. This can be obtained by using $1-u$ in (4.2) in place of u for random variate generation in the second simulation run.

During the simulation, a random sequence of large service times or short interarrival times would cause long

waiting times and conversely. Using the procedure described
above, a sequence of large waiting times would become sequence
of short waiting times in the second run; in other words, the
waiting times in two different runs will be negatively corre-
lated. Let $P_i(t)$ be the probability that waiting times will
exceed t in the ith run, i = 1,2. Using antithetic variates,
$P_1(t)$ and $P_2(t)$ will be negatively correlated; then the esti-
mator will be computed by

$$\widehat{P(T_q > t)} = \frac{1}{2}(P_1(t) + P_2(t))$$

where $P_1(t)$ and $P_2(t)$ are computed according to (4.1).

Generation of antithetic exponential variates for in-
terarrival times is given by (4.3). However, it is not
straightforward to generate antithetic Erlang variates since
the CDF cannot be inverted in closed form. Generating Erlang
variates as sum of k exponential variates with mean β does
not help since equation (4.2) cannot be utilized to obtain
antithetic variates. Nevertheless, one way to solve this
problem is to store the CDF table of chi-square distribution
in an array, then compute $F_x^{-1}(u)$ by linear interpolation to
get a chi-square random variate, and obtain the Erlang
variate by the following transformation

$$E_k = \frac{1}{2} \beta C_k$$

where E_k is the Erland variate with mean kβ and C_k is the
chi-square variate with degrees of freedom k. The tables
for CDF of chi-square distribution are available in [17].

35

3. Data Collection

    a. Selection of Input Parameters

        The input values for each pair of runs were arbitrarily selected to give certain traffic intensity ρ values, usually 0.8 and 0.9 in order to get a wider range of waiting times even though they cause high variability. The width of frequency intervals was chosen to be 0.25 min. so that f_1, for example, would show the number of customers with waiting time less than 0.25 minutes. Then M(t), the number of customers with waiting time greater than t can be computed by

$$M(t) = \sum_{i=4t+1}^{\infty} f_i$$

since t will be in multiples of 0.25. This computation is done in the computer program.

    b. Initial and Final Conditions of the System

        Initial and final conditions of the simulated system are important since they effect the value of parameter estimated. The approach taken in this study was to use epochs. If the time at which the number in the system N changes from 0 to 1 by an arrival is defined as a regeneration point, then the interval between two successive regeneration points can be defined as an epoch [12]. It is obvious that the sequences of waiting times in two different epochs will be independent from each other. The epochs tend to be lengthy with high traffic intensity ρ, large arrival

rate λ, with large number of servers c, or a combination of these three factors. Before each simulation run, the number of epochs for which the simulation is to be run was determined as an input considering those three factors. Experience showed that the first few epochs tend to have waiting times smaller than usual, so they were omitted for data analysis. This isn't feasible for cases with long epochs; however the first couple of epochs have enough information. The simulation runs were started with N = 0 and an arrival so that starting time would be a regeneration point and stopped after the number of epochs determined earlier is completed, i.e. N = 0 was the final condition.

$P_1(t)$ or $P_2(t)$ are given in the output of program for t = 0.25, 0.50, 0.75, ... etc. Then to get the estimators $P(T_q>t)$, all one has to do is to average them for corresponding t values.

B. RESULTS

A comparison of waiting time probabilities obtained from simulation and approximation method is given in Table IV. Investigation of this table indicates that the approximation method agrees quite well with the results of simulation for small t values. The difference between the corresponding values for moderate or large values of t for which the validity of approximation was demonstrated earlier explains the difference between the values when t is small.

TABLE IV.

Comparison of Simulation and Approximation Results

System Parameters				Case	$P(T_q>t)$ Probability that waiting time exceeds t						
c	k	ρ	W_q		t=0.25	t=0.50	t=0.75	t=1.0	t=2.0	t=3.0	t=4.0
3	5	0.6	0.355	S	0.289	0.236	0.188	0.141	0.017	0.005	0.001
				A	0.276	0.215	0.167	0.131	0.048	0.018	0.006
5	5	0.8	1.33	S	0.506	0.467	0.429	0.394	0.259	0.170	0.109
				A	0.499	0.450	0.405	0.365	0.241	0.159	0.105
3	5	0.8	1.942	S	0.592	0.548	0.513	0.475	0.344	0.226	0.147
				A	0.595	0.548	0.504	0.464	0.332	0.238	0.171
4	5	0.8	3.579	S	0.570	0.553	0.536	0.518	0.460	0.398	0.337
				A	0.572	0.549	0.526	0.505	0.427	0.362	0.306
10	15	0.9	1.605	S	0.595	0.551	0.507	0.465	0.313	0.193	0.115
				A	0.603	0.543	0.477	0.441	0.291	0.191	0.126

S: Simulation

A: Approximation

V. CONCLUSION

It was shown in the last chapter that the approximation method developed earlier for the distribution of waiting times can also be used for small values of t. The method for computing steady—state information of $M/E_k/c$ queueing system can now be formally described.

A. DESCRIPTION OF THEPROPOSED COMPUTATION METHOD

After analysis of data and the decision has been made that the particular system under study can be approximated by $M/E_k/c$ system as described in Chapter II, and also having the estimators for λ, k and β obtained (by maximum likelihood or method of moments), one is ready for computations to get the desired information for the system.

1. Delay Probability

First compute service rate μ as

$$\mu = \frac{1}{k\beta} .$$

Then compute traffic intensity ρ and offered load α by

$$\rho = \frac{\lambda}{c\mu} \qquad \text{and} \qquad \alpha = \frac{\lambda}{\mu}$$

respectively. Then enter the chart in Appendix A with α to get delay probability $P(T_q>0)$. Use the charts in [10] or [11] if number of servers c>16.

39

2. **Average Waiting Time and Average Queue Length**

First compute a system constant, namely b, by

$$b = \frac{2c\mu(1-\rho)}{1 + \frac{1}{k}}$$

Then average waiting time W_q is computed as follows

$$W_q = \frac{P(T_q > 0)}{b}$$

Also, average queue length L_q is given by

$$L_q = \lambda W_q .$$

3. **Average Time in System and Average Number in System**

Average time in system W and average number in system L will be computed by

$$W = W_q + \frac{1}{\mu} \qquad \text{and} \qquad L = L_q + \frac{\lambda}{\mu}$$

respectively.

4. **Waiting Time Distribution**

The CDF of waiting times is given by

$$F_{Tq}(t) = 1 - P(T_q > 0)e^{-bt} , \qquad t \geq 0$$

where b is the system constant computed in step 2.

Then the following probabilities can be computed

(i) $P(T_q > t_1) = P(T_q > 0)e^{-bt_1}$

(ii) $P(t_1 < T_q < t_2) = P(T_q > 0)(e^{-bt_1} - e^{-bt_2})$

(iii) $P(T_q < t_1) = 1 - P(T_q > 0)e^{-bt_1}$

as needed by the user.

B. DISCUSSION

1. Advantages of the Method

(i) Needless to say, the most important advantage of the method is simplicity. The kind of information listed above can be computed in minutes given c, λ, k and μ which would be needed to compute anyway. All of the computations can be laid down on one regular size page so that it is very easy to follow by somebody else.

(ii) In most cases, one has to compute the above listed information to determine the effect of changing the number of servers c on the selected system variables. This multiplies the savings of time and computational effort.

(iii) No tables are necessary except for the delay probability chart. No interpolation would be needed.

2. Disadvantages of the Method

(i) Since the method gives an approximation, it is not too precise even though the results are almost always in $\mp$ 5 percent of the actual value, and in $\mp$ 2 percent for most cases.

(ii) The approximation of state probabilities with this method does not appear to be direct.

3. Applications

One of the characteristic applications of the queueing theory is to investigate the effect of changing the parameters or number of servers on the measure of effectiveness (MOE) assigned by the management. This measure of effectiveness can be delay probability, average waiting time

or probability of waiting time exceeding time t etc. The
marginal utility of c + 1st server will be the difference
between the values of MOE's for c and c+1 servers. Then the
decision criterion whether or not to hire the c+1st server
is how his cost compares to his marginal utility.

Another managerial objective may be to require, for
example, the probability of waiting time exceeds 4 min.,
$P(T_q > 4) = 0.05$. The number of servers necessary to achieve
this purpose will then be of interest.

It is possible to extend the variety of examples.
The common property of the above examples is that precision
greater than that of suggested approximation is not needed
to make the decision.

4. Final Remarks

The author wishes to emphasize that without the
numerical tables provided by Hillier and Lo [6], it would
have been impossible to develop such an approximation method.
The method may need some modifications as new tables for
the cases not covered in [6] become available.

APPENDIX A

CHARTS FOR DELAY PROBABILITY

In the two following charts, the delay probabilities can be found for M/M/c system which were shown to be very close to those of $M/E_k/c$ system, with number of servers c up to about 16. One has to enter to chart with offered load α as abscissa, then delay probability $P(T_q > 0)$ is the ordinate value of the point on the proper c curve which has α as abscissa value.

Delay Probability $P(T_q > 0)$ vs Offered Load α

44

Delay Probability P($T_q>0$)

Offered Load α

45
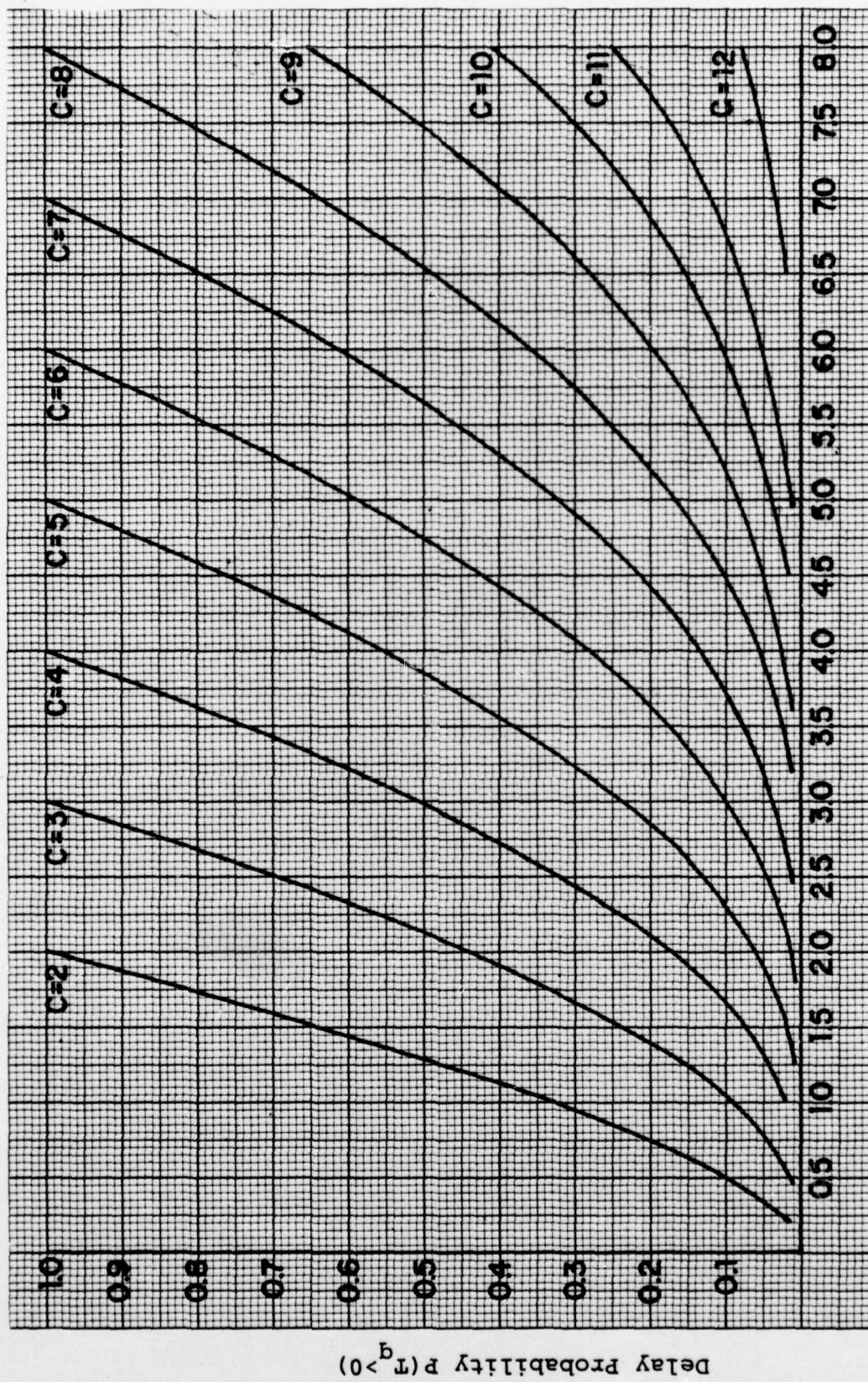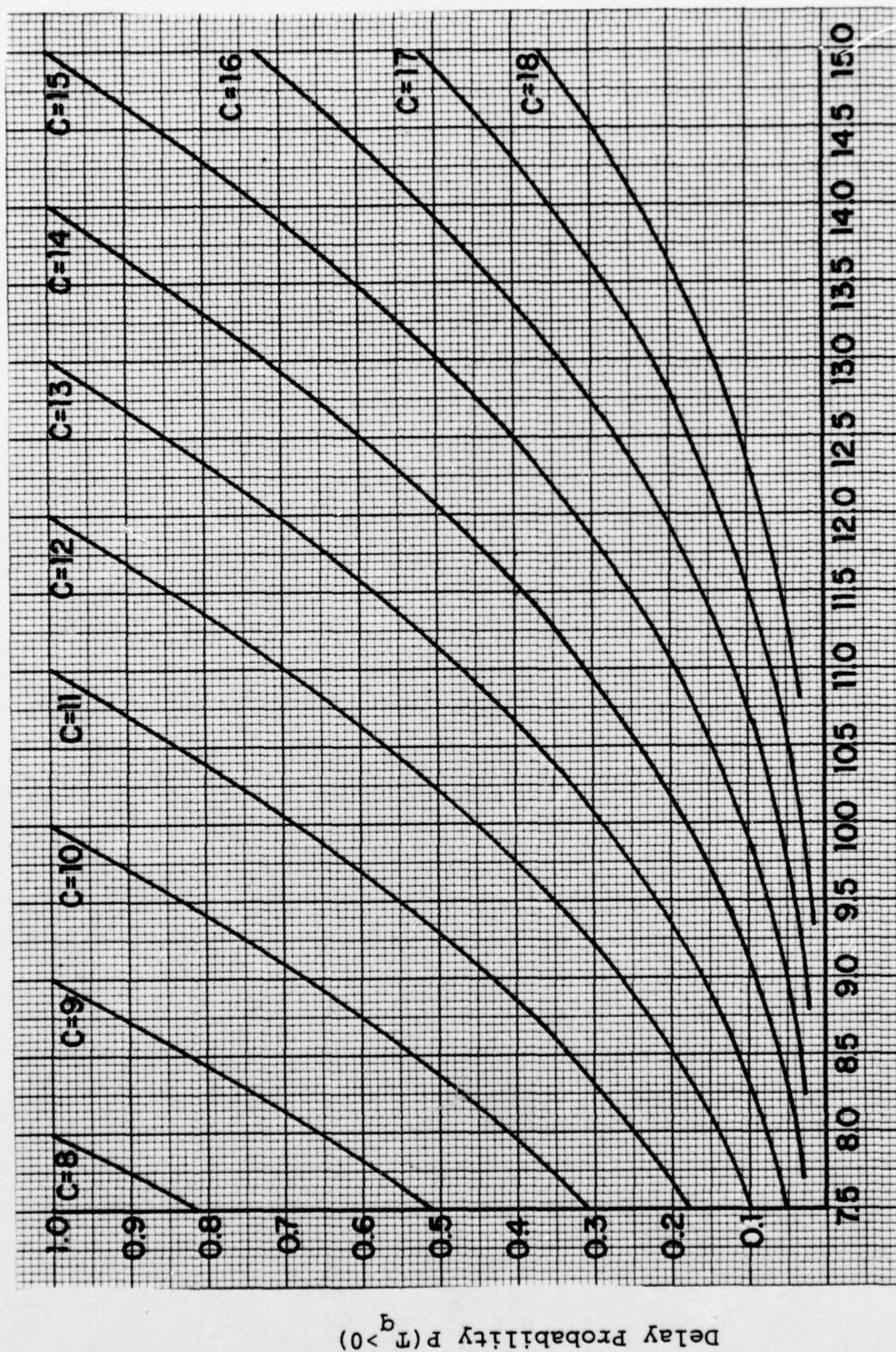
APPENDIX B

SAMPLE PAGES FROM NUMERICAL TABLES

N = 1 , K = 8 , C = 2 , RHO = 0.10

STATE I	P(N=I)	P(N<=I)	STATE I	P(N=I)	P(N<=I)
0	0.8179210	0.8179210	7	5.599922'-09	0.9999999
1	0.1641578	0.9820789	8	2.117447'-10	0.9999999
2	0.01666797	0.9987469	9	7.714040'-12	0.9999999
3	0.001182842	0.9999297	10	2.749428'-13	0.9999999
4	6.689144'-05	0.9999966	11	9.683172'-15	0.9999999
5	3.236087'-06	0.9999998	12	3.395385'-16	0.9999999
6	1.400809'-07	0.9999999	13	1.190273'-17	0.9999999

N = 1 , K = 8 , C = 2 , RHO = 0.20

STATE I	P(N=I)	P(N<=I)	STATE I	P(N=I)	P(N<=I)
0	0.6652663	0.6652663	9	6.807781'-09	0.9999999
1	0.2694672	0.9347336	10	5.887453'-10	0.9999999
2	0.05593797	0.9906716	11	5.088657'-11	0.9999999
3	0.008232649	0.9989042	12	4.402216'-12	0.9999999
4	0.0009825202	0.9998868	13	3.812238'-13	0.9999999
5	0.0001024607	0.9999892	14	3.303549'-14	1.000000
6	9.799626'-06	0.9999990	15	2.863789'-15	1.000000
7	8.890805'-07	0.9999999	16	2.482840'-16	1.000000
8	7.833808'-08	0.9999999	17	2.152598'-17	1.000000

N = 1 , K = 8 , C = 2 , RHO = 0.30

STATE I	P(N=I)	P(N<=I)	STATE I	P(N=I)	P(N<=I)
0	0.5352489	0.5352489	9	4.448650'-07	0.9999999
1	0.3295022	0.8647510	10	6.724838'-08	0.9999999
2	0.1053992	0.9701503	11	1.017040'-08	0.9999999
3	0.02428844	0.9944387	12	1.538943'-09	0.9999999
4	0.004626255	0.9990650	13	2.329388'-10	0.9999999
5	0.0007867496	0.9998517	14	3.526286'-11	0.9999999
6	0.0001254372	0.9999772	15	5.338370'-12	0.9999999
7	1.934617'-05	0.9999965	16	8.081644'-13	0.9999999
8	2.940745'-06	0.9999995	17	1.223453'-13	0.9999999

N = 1 , K = 8 , C = 2 , RHO = 0.40

STATE I	P(N=I)	P(N<=I)	STATE I	P(N=I)	P(N<=I)
0	0.4233869	0.4233869	11	5.045168'-07	0.9999998
1	0.3532261	0.7766131	12	1.155876'-07	0.9999999
2	0.1553454	0.9319585	13	2.648409'-08	0.9999999
3	0.05005832	0.9820168	14	6.068323'-09	0.9999999
4	0.01359216	0.9956089	15	1.390444'-09	0.9999999
5	0.003360750	0.9989697	16	3.185936'-10	0.9999999
6	0.0007926517	0.9997624	17	7.299952'-11	0.9999999
7	0.0001831328	0.9999455	18	1.672643'-11	0.9999999
8	4.200506'-05	0.9999875	19	3.832525'-12	0.9999999
9	9.618429'-06	0.9999971	20	8.781472'-13	0.9999999
10	2.202535'-06	0.9999993	21	2.012101'-13	0.9999999

N = 1 , K = 8 , C = 2 , RHO = 0.50

STATE I	P(N=I)	P(N<=I)	STATE I	P(N=I)	P(N<=I)
0	0.3265101	0.3265101	12	3.613901'-06	0.9999983
1	0.3469798	0.6734899	13	1.157263'-06	0.9999994
2	0.1972846	0.8707746	14	3.705567'-07	0.9999998
3	0.08362406	0.9543986	15	1.186718'-07	0.9999999
4	0.03042698	0.9848256	16	3.800184'-08	0.9999999
5	0.01025498	0.9950806	17	1.216919'-08	0.9999999
6	0.003359760	0.9984204	18	3.896893'-09	0.9999999
7	0.001073734	0.9994941	19	1.247838'-09	0.9999999
8	0.0003439242	0.9998381	20	3.996063'-10	0.9999999
9	0.0001100940	0.9999481	21	1.279645'-10	0.9999999
10	3.524644'-05	0.9999834	22	4.097760'-11	0.9999999
11	1.128574'-05	0.9999946	23	1.312210'-11	0.9999999

N = 1 , K = 8 , C = 2 , RHO = 0.55

STATE I	P(N=I)	P(N<=I)	STATE I	P(N=I)	P(N<=I)
0	0.2829307	0.2829307	12	1.603995'-05	0.9999905
1	0.3341386	0.6170692	13	5.949960'-06	0.9999965
2	0.2127340	0.8298033	14	2.207117'-06	0.9999987
3	0.1018512	0.9316546	15	8.187226'-07	0.9999995
4	0.04223055	0.9738851	16	3.037022'-07	0.9999998
5	0.01634198	0.9902267	17	1.126572'-07	0.9999999
6	0.006141469	0.9963682	18	4.178976'-08	0.9999999
7	0.002284580	0.9986528	19	1.550175'-08	0.9999999
8	0.0008475561	0.9995003	20	5.750316'-09	0.9999999
9	0.0003143165	0.9998146	21	2.133059'-09	0.9999999
10	0.0001165765	0.9999312	22	7.912502'-10	0.9999999
11	4.324122'-05	0.9999745	23	2.935112'-10	0.9999999

N = 1 , K = 8 , C = 2 , RHO = 0.60

STATE I	P(N=I)	P(N<=I)	STATE I	P(N=I)	P(N<=I)
0	0.2422594	0.2422594	13	2.673833'-05	0.9999802
1	0.3154812	0.5577406	14	1.136478'-05	0.9999915
2	0.2231736	0.7809142	15	4.834706'-06	0.9999964
3	0.1197532	0.9006675	16	2.055853'-06	0.9999985
4	0.05612541	0.9567929	17	8.741895'-07	0.9999993
5	0.02471780	0.9815107	18	3.717262'-07	0.9999997
6	0.01061813	0.9921288	19	1.580669'-07	0.9999999
7	0.004524238	0.9966531	20	6.721392'-08	0.9999999
8	0.001923908	0.9985770	21	2.858242'-08	0.9999999
9	0.0008179441	0.9993949	22	1.215429'-08	0.9999999
10	0.0003477749	0.9997427	23	5.167870'-09	0.9999999
11	0.0001478774	0.9998906	24	2.197502'-09	0.9999999
12	6.288060'-05	0.9999534	25	9.344305'-10	0.9999999

N = 1 , K = 8 , C = 2 , RHO = 0.65

STATE I	P(N=I)	P(N<=I)	STATE I	P(N=I)	P(N<=I)
0	0.2042829	0.2042829	13	0.0001066369	0.9999002
1	0.2914341	0.4957170	14	5.154022'-05	0.9999518
2	0.2275953	0.7233124	15	2.490176'-05	0.9999767
3	0.1359851	0.8592975	16	1.203133'-05	0.9999887
4	0.07154649	0.9308441	17	5.812964'-06	0.9999945
5	0.03559288	0.9664370	18	2.808544'-06	0.9999973
6	0.01733483	0.9837718	19	1.356953'-06	0.9999987
7	0.008387748	0.9921595	20	6.556148'-07	0.9999993
8	0.004052620	0.9962122	21	3.167615'-07	0.9999997
9	0.001957785	0.9981700	22	1.530440'-07	0.9999998
10	0.0009458477	0.9991158	23	7.394365'-08	0.9999999
11	0.0004569793	0.9995728	24	3.572594'-08	0.9999999
12	0.0002207896	0.9997936	25	1.726136'-08	0.9999999

N = 1 , K = 8 , C = 2 , RHO = 0.70

STATE I	P(N=I)	P(N<=I)	STATE I	P(N=I)	P(N<=I)
0	0.1688107	0.1688107	14	0.0002079035	0.9997511
1	0.2623786	0.4311892	15	0.0001132721	0.9998644
2	0.2249728	0.6561621	16	6.171406'-05	0.9999261
3	0.1488435	0.8050057	17	3.362367'-05	0.9999597
4	0.08739370	0.8923994	18	1.831919'-05	0.9999781
5	0.04879244	0.9411919	19	9.980949'-06	0.9999880
6	0.02675223	0.9679441	20	5.437867'-06	0.9999934
7	0.01459134	0.9825354	21	2.962714'-06	0.9999964
8	0.007949822	0.9904853	22	1.614175'-06	0.9999980
9	0.004330918	0.9948162	23	8.794515'-07	0.9999989
10	0.002359518	0.9971757	24	4.791517'-07	0.9999994
11	0.001285522	0.9984612	25	2.610562'-07	0.9999996
12	0.0007003895	0.9991616	26	1.422313'-07	0.9999998
13	0.0003815931	0.9995432	27	7.749165'-08	0.9999999

N = 1 , K = 8 , C = 2 , RHO = 0.75

STATE I	P(N=I)	P(N<=I)	STATE I	P(N=I)	P(N<=I)
0	0.1356713	0.1356713	16	0.0002794797	0.9995621
1	0.2286572	0.3643286	17	0.0001705845	0.9997327
2	0.2142547	0.5785834	18	0.0001041197	0.9998369
3	0.1562144	0.7347978	19	6.355041'-05	0.9999004
4	0.1018311	0.8366290	20	3.878892'-05	0.9999392
5	0.06344114	0.9000702	21	2.367538'-05	0.9999629
6	0.03691755	0.9369877	22	1.445062'-05	0.9999774
7	0.02577329	0.9627610	23	8.820148'-06	0.9999862
8	0.01451033	0.9772713	24	5.383507'-06	0.9999915
9	0.008856043	0.9861274	25	3.285902'-06	0.9999948
10	0.005405270	0.9915327	26	2.005598'-06	0.9999968
11	0.003299163	0.9948318	27	1.224146'-06	0.9999980
12	0.002013691	0.9968455	28	7.471757'-07	0.9999988
13	0.001229035	0.9980746	29	4.560496'-07	0.9999993
14	0.0007501903	0.9988248	30	2.783565'-07	0.9999995
15	0.0004578896	0.9992827	31	1.698990'-07	0.9999997

N = 1 , K = 8 , C = 2 , RHO = 0.80

STATE I	P(N=I)	P(N<=I)	STATE I	P(N=I)	P(N<=I)
0	0.1047099	0.1047099	19	0.0003509128	0.9992547
1	0.1905802	0.2952901	20	0.0002385763	0.9994933
2	0.1943588	0.4896489	21	0.0001621989	0.9996555
3	0.1555139	0.6451628	22	0.0001102738	0.9997658
4	0.1120363	0.7571992	23	7.497263'-05	0.9998407
5	0.07749480	0.8346940	24	5.097079'-05	0.9998917
6	0.05289866	0.8875927	25	3.465339'-05	0.9999264
7	0.03598627	0.9235789	26	2.355971'-05	0.9999499
8	0.02446579	0.9480447	27	1.601766'-05	0.9999660
9	0.01663278	0.9646775	28	1.088977'-05	0.9999768
10	0.01130789	0.9759854	29	7.403608'-06	0.9999843
11	0.007687837	0.9836733	30	5.033474'-06	0.9999893
12	0.005226709	0.9888999	31	3.422096'-06	0.9999927
13	0.003553671	0.9924535	32	2.326573'-06	0.9999951
14	0.002415890	0.9948694	33	1.581761'-06	0.9999966
15	0.001642486	0.9965118	34	1.075380'-06	0.9999977
16	0.001116673	0.9976285	35	7.311221'-07	0.9999984
17	0.0007591899	0.9983877	36	4.970663'-07	0.9999989
18	0.0005161487	0.9989038	37	3.379391'-07	0.9999992

N = 1 , K = 8 , C = 2 , RHO = 0.85

STATE I	P(N=I)	P(N<=I)	STATE I	P(N=I)	P(N<=I)
0	0.07578546	0.07578546	25	0.0003038200	0.9990715
1	0.1484290	0.2242145	26	0.0002289153	0.9993004
2	0.1641668	0.3883814	27	0.0001724776	0.9994729
3	0.1436229	0.5320042	28	0.0001299546	0.9996028
4	0.1138889	0.6458931	29	9.791527'-05	0.9997007
5	0.08706796	0.7329611	30	7.377495'-05	0.9997745
6	0.06581450	0.7987756	31	5.558625'-05	0.9998301
7	0.04961175	0.8483874	32	4.188185'-05	0.9998720
8	0.03738017	0.8857675	33	3.155618'-05	0.9999036
9	0.02816350	0.9139310	34	2.377623'-05	0.9999273
10	0.02121974	0.9351507	35	1.791437'-05	0.9999452
11	0.01598812	0.9511389	36	1.349770'-05	0.9999587
12	0.01204636	0.9631853	37	1.016996'-05	0.9999689
13	0.009076413	0.9722617	38	7.662616'-06	0.9999766
14	0.006838687	0.9791003	39	5.773451'-06	0.9999823
15	0.005152650	0.9842530	40	4.350048'-06	0.9999866
16	0.003882306	0.9881353	41	3.277574'-06	0.9999899
17	0.002925151	0.9910604	42	2.469511'-06	0.9999924
18	0.002203976	0.9932644	43	1.860671'-06	0.9999943
19	0.001660601	0.9949250	44	1.401935'-06	0.9999957
20	0.001251192	0.9961762	45	1.056296'-06	0.9999967
21	0.0009427192	0.9971189	46	7.958755'-07	0.9999976
22	0.0007102983	0.9978293	47	5.996580'-07	0.9999982
23	0.0005351794	0.9983644	48	4.518166'-07	0.9999986
24	0.0004032349	0.9987677	49	3.404244'-07	0.9999989

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$$M = 1 \quad , \quad K = 8 \quad , \quad C = 2$$

RHO	P(DELAY)	L(GIVEN K)	LQ(GIVEN K)	LQ FOR K=1	RATIO
0.10	0.01792109	0.2013269	0.001326923	0.002020202	0.6568266
0.20	0.06526637	0.4105492	0.01054924	0.01666667	0.6329545
0.30	0.1352489	0.6365210	0.03652108	0.05934066	0.6154479
0.40	0.2233869	0.8917542	0.09175420	0.1523809	0.6021371
0.50	0.3265101	1.197244	0.1972441	0.3333333	0.5917324
0.55	0.3829307	1.380202	0.2802032	0.4770609	0.5873531
0.60	0.4422594	1.593808	0.3938082	0.6750000	0.5834197
0.65	0.5042829	1.851505	0.5515060	0.9510822	0.5798720
0.70	0.5688107	2.175664	0.7756644	1.345098	0.5766602
0.75	0.6356713	2.606503	1.106503	1.928571	0.5737424
0.80	0.7047099	3.224415	1.624415	2.844444	0.5710835
0.85	0.7757854	4.216932	2.516932	4.426126	0.5686536
0.90	0.8487694	6.146583	4.346582	7.673684	0.5664272
0.95	0.9235438	11.82589	9.925892	17.58717	0.5643823
0.98	0.9692222	28.73332	26.77332	47.53494	0.5632344
0.99	0.9845791	56.86909	54.88910	97.51749	0.5628642

COMPUTER PROGRAM

```
PREAMBLE
PERMANENT ENTITIES
THE SYSTEM OWNS A QUEUE
EVERY SERVER HAS A STATE,AN IDENTI
TEMPORARY ENTITIES
EVERY JOB HAS AN ARR.TIME AND MAY BELONG TO THE QUEUE
EVENT NOTICES INCLUDE ARRIVAL,END.OF.SIMULATION
EVERY DEPARTURE HAS A CODE
PRIORITY ORDER IS ARRIVAL,DEPARTURE,END.OF.SIMULATION
NORMALLY,MODE IS REAL
TALLY AV.Q.T AS THE MEAN AND VAR.Q AS THE VARIANCE OF R
TALLY FREQ(0 TO 15 BY 0.25) AS THE HISTOGRAM OF R
TALLY AVER AS THE MEAN AND VARER AS THE VARIANCE OF ERLG
ACCUMULATE MMM AS THE MEAN OF K
ACCUMULATE KKK AS THE SUM OF MM
ACCUMULATE FRES(0 TO 40 BY 1) AS THE HISTOGRAM OF K
DEFINE FREE TO MEAN 0
DEFINE BUSY TO MEAN 1
DEFINE X,Y,XX AND YY AS REAL,1-DIMENSIONAL ARRAYS
DEFINE F,Z AS REAL,1-DIMENSIONAL ARRAYS
DEFINE NN,KK,STP,EPSTP,ST.N,EP AS INTEGER VARIABLES
DEFINE U,ERLG,TOT1,AVG,MM AS REAL VARIABLES
DEFINE STS1,STS2,STS3 AS INTEGER VARIABLES
DEFINE SD1,SD2,SD3 AS INTEGER VARIABLES
DEFINE FLAG AS AN INTEGER VARIABLE
DEFINE LAMBDA,MU,S,TOT.Q.TIME,R,SUMSQR,D,BETA,ST,STT,START.T
       AS REAL VARIABLES
DEFINE CUSTOMER,J,K,DELAYER,AR,M,N,ALPHA#CA2E.NO,I,EFLAG
       AS INTEGER VARIABLES
END



MAIN
READ CASE.NO
READ J
''  J IS THE NUMBER OF SERVERS
READ ALPHA
''  ALPHA IS SHAPE PARAMETER OF THE SERVICE TIME DISTRIBUTION
READ BETA
''  BETA IS SCALE PARAMETER OF THE SERVICE TIME DISTRIBUTION
READ LAMBDA
''  LAMBDA IS ARRIVAL RATE
READ EPSTP
''  EPSTP IS TOTAL NUMBER OF EPOCHS FOR THIS SIMULATION RUN
READ FLAG
''  FLAG INDICATES WHICH PAIR OF ANTITHETIC VARIATES WILL BE
''                                                       USED
READ SD1,SD2,SD3
READ NN,KK
''  NN IS DIMENSION OF ARRAY F
''  KK IS NUMBER OF ELEMENTS NE 1 IN ARRAY F
RESERVE F(*) AS NN AND Z(*) AS NN
''  READ CDF TABLE OF THE CHI-SQR DISTN. WITH 2*ALPHA
''                        DEGREES OF FREEDOM
FOR I=1 TO KK,DO
   READ F(I)
   LET F(I)=1.0-F(I)
LOOP
FOR I=1 TO KK,DO
   READ Z(I)
LOOP
FOR I=KK+1 TO NN,DO
   LET F(I)=1.0
LOOP
```

```
LET N.SERVER=J
CREATE EACH SERVER
LET K=0
LET STP=0
LET STS1=SEED.V(SD1)
LET STS2=SEED.V(SD2)
LET STS3=SEED.V(SD3)
START NEW PAGE
SCHEDULE AN ARRIVAL NOW
START SIMULATION
STOP
END



EVENT ARRIVAL SAVING THE EVENT NOTICE
DEFINE JJ AS AN INTEGER VARIABLE
ADD 1 TO AR
LET T=TIME.V*1440
IF STP EQ 1
DESTROY THIS ARRIVAL
RETURN
OTHERWISE
ADD 1 TO K
'' GENERATE INTERARRIVAL TIME
LET U=UNIFORM.F(0.0,1.0,SD1)
IF FLAG EQ 1
LET U=1.0-U
ALWAYS
LET AR.TIME=-LOG.E.F(U)/LAMBDA
RESCHEDULE THIS ARRIVAL IN AR.TIME MINUTES
LET JJ=0
FOR EVERY SERVER,DO
IF STATE(SERVER)=BUSY
ADD 1 TO JJ
ALWAYS
LOOP
IF JJ=0
   LET EFLAG=1
   CALL EPOCH
   GO TO AA
ALWAYS
LET J1=J-1
IF JJ EQ J1
   LET MM=1.0
ALWAYS
'AA'
'NEXT'   FOR EVERY SERVER,DO
IF STATE(SERVER)=FREE
   LET STATE(SERVER)=BUSY
LET R=0.0
'ADD'   ADD 1 TO CUSTOMER
'' GENERATE SERVICE TIME
LET U=UNIFORM.F(0.0,1.0,SD2)
IF FLAG EQ 1
LET U=1.0-U
ALWAYS
CALL ERLNG
LET DEP.TIME=ERLG
LET IDENTI(SERVER)=DEP.TIME
SCHEDULE A DEPARTURE GIVEN DEP.TIME IN DEP.TIME MINUTES
RETURN
OTHERWISE
LOOP
CREATE A JOB
LET ARR.TIME(JOB)=TIME.V*1440
FILE THIS JOB IN THE QUEUE
RETURN
END
```

```
EVENT DEPARTURE SAVING THE EVENT NOTICE
DEFINE JJ AS AN INTEGER VARIABLE
LET T=TIME.V*1440
IF STP EQ 1
DESTROY THIS DEPARTURE
RETURN
OTHERWISE
SUBTRACT 1 FROM K
LET JJ=0
FOR EACH SERVER,DO
   IF STATE(SERVER)=BUSY
      ADD 1 TO JJ
   ALWAYS
LOOP
IF JJ EQ 1
   LET EFLAG=0
   CALL EPOCH
   GO TO AA
OTHERWISE
IF JJ EQ J
   IF N.QUEUE EQ 0
      LET MM=0.0
   ALWAYS
ALWAYS
'AA'
FOR EVERY SERVER,DO
      IF IDENTI(SERVER)=CODE(DEPARTURE)
IF THE QUEUE IS EMPTY
      LET STATE(SERVER)=FREE
DESTROY THE DEPARTURE
RETURN
OTHERWISE
LET A.T=ARR.TIME(F.QUEUE)
REMOVE FIRST JOB FROM THE QUEUE
DESTROY THIS JOB
LET R=T-A.T
ADD 1 TO CUSTOMER
ADD 1 TO DELAYER
ADD R TO TOT.Q.TIME
ADD R**2 TO SUMSQR
'' GENERATE SERVICE TIME
LET U=UNIFORM.F(0.0,1.0,SD3)
IF FLAG EQ 1
LET U=1.0-U
ALWAYS
CALL ERLNG
LET DEP.TIME=ERLG
LET IDENTI(SERVER)=DEP.TIME
SCHEDULE A DEPARTURE GIVEN DEP.TIME IN DEP.TIME MINUTES
RETURN
OTHERWISE
LOOP
END




EVENT END.OF.SIMULATION
DEFINE FR,LUCKY AS INTEGER VARIABLES
DEFINE DEL1 AS AN INTEGER VARIABLE
RESERVE X(*) AS 60,Y(*) AS 60 AND YY(*) AS 60
RESERVE XX(*) AS 60
LET S=TIME.V*1440
START NEW PAGE
LET MU=1.0/(ALPHA*BETA)
'' MU IS SERVICE RATE
LET G=LAMBDA/MU
'' G IS OFFERED LOAD
LET RO=G/J
```

```
''  RO IS TRAFFIC INTENSITY
''  COMPUTE AVG.WAIT.TIME FOR EQUIVALENT M/M/C QUEUE
LET JFAC=1
FOR I=1 TO J-1,DO
    LET JFAC=JFAC*(I+1)
LOOP
LET C=((J*RO)**J)/(JFAC*(1.0-RO))
LET B=1.0
FOR K=1 TO J-1,DO
LET KFAC=1
    FOR L=1 TO K-1,DO
    LET KFAC=KFAC*(L+1)
    LOOP
ADD ((J*RO)**K)/KFAC TO B
LOOP
LET PONE=B+C
LET PNOT=1.0/(B+C)
LET E=PNOT*((J*RO)**J)/(J*JFAC*((1.0-RO)**2)*MU)
LET VSQR=1.0/ALPHA
LET V=SQRT.F(VSQR)
LET W=(E/2.0)*(1.0+VSQR)
LET PMMC=E*J*MU*(1.0-RO)
PRINT 1 LINE WITH CASE.NO AND FLAG AS FOLLOWS
        CASE NO=****                    FLAG=*
SKIP 2 LINES
PRINT 2 LINES AS FOLLOWS
INPUT INFORMATION:

SKIP 2 LINES
PRINT 5 LINES WITH STS1,SEED.V(SD1),STS2,SEED.V(SD2),
    STS3,SEED.V(SD3) AS FOLLOWS
        STARTING SEEDS:              FINAL SEEDS:

        I.  **********              **********
        II. **********              **********
        III. **********             **********
SKIP 2 LINES
PRINT 1 LINE WITH ALPHA AND BETA AS FOLLOWS
        ALPHA=***       BETA=**.****
SKIP 2 LINES
PRINT 1 LINE WITH LAMBDA AS FOLLOWS
        LAMBDA=***.****
SKIP 2 LINES
PRINT 1 LINE WITH MU AS FOLLOWS
        MU=**.****
SKIP 2 LINES
PRINT 1 LINE WITH RO AS FOLLOWS
TRAFFIC INTENSITY,RO=*.****
SKIP 2 LINES
PRINT 1 LINE WITH J AS FOLLOWS
NUMBER OF SERVERS=**
SKIP 2 LINES
PRINT 1 LINE WITH G AS FOLLOWS
OFFERED LOAD=**.****
SKIP 2 LINES
PRINT 1 LINE WITH V AS FOLLOWS
COEFFICIENT OF VARIATION=**.****
START NEW PAGE
IF EFLAG=1
ADD S-START.T TO SUMP
ALWAYS
LET LUCKY=CUSTOMER-DELAYER
LET PD=DELAYER/CUSTOMER
PRINT 2 LINES AS FOLLOWS
OUTPUT INFORMATION:

SKIP 2 LINES
PRINT 1 LINE WITH AV.Q.T AS FOLLOWS
MEAN QUEUEING TIME FROM SIMULATION=***.****
SKIP 2 LINES
PRINT 1 LINE WITH W AS FOLLOWS
COMPUTED MEAN QUEUEING TIME=***.**** MINUTES
```

```
SKIP 2 LINES
PRINT 1 LINE WITH W-AV.Q.T AS FOLLOWS
DIFFERENCE=**.****
SKIP 2 LINES
PRINT 1 LINE WITH VAR.Q AS FOLLOWS
VARIANCE OF QUEUEING-TIME=***.****
SKIP 2 LINES
PRINT 1 LINE WITH MMM AS FOLLOWS
AVERAGE QUEUE LENGTH=**.****
SKIP 2 LINES
PRINT 1 LINE WITH CUSTOMER AS FOLLOWS
TOTAL NUMBER SERVED IN SIMULATION PERIOD=*****
SKIP 2 LINES
PRINT 1 LINE WITH DELAYER AS FOLLOWS
TOTAL NUMBER OF CUSTOMERS WITH DELAY GREATER THAN ZERO=*****
LET PERIOD=S-STT
SKIP 2 LINES
PRINT 1 LINE WITH PD AS FOLLOWS
PROB(W>0)=*.******
SKIP 2 LINES
PRINT 1 LINE WITH PMMC AS FOLLOWS
PMMC=*.******
SKIP 2 LINES
PRINT 1 LINE WITH PERIOD AS FOLLOWS
SIMULATION TIME=*******.****   MINUTES
SKIP 2 LINES
PRINT 1 LINE WITH KKK*1440 AS FOLLOWS
TOTAL BUSY PERIOD=********.****   MINUTES
FOR I=1 TO 60,DO
   LET XX(I)=FREQ(I)
LOOP
SUBTRACT LUCKY FROM XX(1)
'' COMPUTE AND PRINT F SUB I
FOR I=1 TO N,DO
ADD XX(I) TO FR
LET Y(I)=1.0-(FR/DELAYER)
IF FLAG EQ 0
IF FR EC DELAYER
LET N=I-1
GO TO REG
OTHERWISE
ALWAYS
LET Y(I)=LOG.E.F(Y(I))
LET X(I)=I/4.0
LOOP
'REG'
LET CUMFREQ=0
START NEW PAGE
'' COMPUTE AND PRINT STATE PROBABILITIES
PRINT 2 LINES AS FOLLOWS
I               X(I)                FREQ(I)                  CUMFREQ

FOR I=1 TO N,DO
ADD XX(I) TO CUMFREQ
   SKIP 1 LINE
PRINT 1 LINE WITH I,X(I),XX(I) AND CUMFREQ AS FOLLOWS
 **             **.***           ********               ********
LOOP
IF FLAG EQ 1
   CALL REGRESSION
ALWAYS
START NEW PAGE
PRINT 2 LINES AS FOLLOWS
 N                    T(N)                     P(N)

FOR I=1 TO 40,DO
PRINT 1 LINE WITH I,FRES(I)*1440.0 AND (FRES(I)*1440.)/PERIO
                      AS FOLLOWS
 **              *****.****                 *.********
LOOP
RETURN
END
```

```
ROUTINE EPOCH
DEFINE FI.N AS AN INTEGER VARIABLE
LET T=TIME.V*1440
IF EFLAG EQUALS 1
'' EPOCH STARTS
LET START.T=T
LET ST.N=CUSTOMER
LET TOT1=TOT.Q.TIME
ADD 1 TO EP
RETURN
OTHERWISE
'' EPOCH ENDS
LET FINISH.T=T
LET FI.N=CUSTOMER
LET TOT2=TOT.Q.TIME
LET AVG=(TOT2-TOT1)/((FI.N-ST.N)*1.0)
LET PERIOD=FINISH.T-START.T
LET CUM.AVG=TOT2/FI.N
SKIP 1 LINE
LIST EP,START.T,FINISH.T,ST.N,FI.N,TOT1,TOT2
LIST PERIOD,TOT2-TOT1,FI.N-ST.N,AVG,CUM.AVG
'' CHECK FOR END CF SIMULATION
IF EP EQ EPSTP
   LET STP=1
   SCHEDULE AN END.OF.SIMULATION NOW
ALWAYS
RETURN
END




ROUTINE ERLNG
DEFINE RL,LH,RH AS INTEGER VARIABLES
IF U LT F(1)
   LET CHSQR=Z(1)*(U/F(1))
GO TO ERLANG
ALWAYS
'' START BISECTION SEARCH
IF U GT F(KK)
LET CHSQR=Z(KK)+(Z(KK)-Z(KK-1))*(U-F(KK))
   GO TO ERLANG
ALWAYS
LET LH=0
LET RH=NN
'AA' LET RL=(LH+RH)/2
   IF U LT F(RL)
      LET RH=RL
      GO TO BB
      ALWAYS
   LET LH=RL
'BB' IF RH=LH+1
         GO TO QUIT
      ALWAYS
      GO TO AA
'QUIT'
'' DO LINEAR INTERPOLATION
LET CHSQR=Z(LH)+((U-F(LH))*(Z(RH)-Z(LH)))/(F(RH)-F(LH))
'ERLANG'
LET ERLG=(BETA*CHSQR)/2.0
RETURN
END
```

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